Efficient Generation of Trial Trajectories for Use in the Monte Carlo Simulation of Energetic Particle Telescopes

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Abstract: The efficiency of Monte Carlo simulations of the response of detector telescopes for energetic particle studies can sometimes be significantly increased if trial trajectories are required to pass through two different detectors that will be required in coincidence for selecting the particles to be analyzed. However the distribution of trajectories must correspond to an isotropic or other assumed distribution of incidence directions in the particle population to be measured. Adopting the approximation that particle trajectories through the instrument are straight lines, we present a simple technique for generating the needed angular distribution starting from line segments whose end points are generated uniformly on two non-coplanar polygonal surfaces. A subset of these lines is discarded on a statistical basis to yield the required angular distribution.

Keywords: energetic particle telescopes, Monte Carlo simulation

1 Introduction

Derivation of energetic particle intensities from count rates measured using multi-detector instruments (often called “particle telescopes”) requires the calculation of the instrument’s collecting power. The calculation can depend on both the instrument geometry and the response of the detectors that make it up. The problem of calculating the collecting power in the case where only the instrument geometry plays a role was discussed by Sullivan[1], who presented analytical results for a number of simple detector geometries and also outlined an approach for using Monte Carlo techniques to treat more complex geometries. Following the usual convention, we use the term “geometrical factor” to denote the collecting power when it is limited only by the geometry of the detector system.

In many cases particle trajectories can be assumed to be straight lines, which significantly simplifies the calculation. In the following discussion we adopt this assumption. Thus we are interested in the following problems: given two finite plane surfaces $S_1$ and $S_2$ how can one 1) calculate the geometrical factor for trajectories sampled from an isotropic distribution of incidence directions and required to pass through both surfaces and 2) efficiently generate a set of trajectories passing through both surfaces that correctly reflects the incident isotropic distribution. In addition, we are interested in generalizing these results to the case of an arbitrary distribution of incidence directions. This latter topic is addressed in Section 4.

A simple, commonly employed technique for generating a set of trajectories drawn from an isotropic distribution so that they pass through two given surfaces, $S_1$ and $S_2$, is to randomly generate $N$ points on one of the surfaces, say $S_1$, with a statistically uniform density per unit area. For each of these points a polar angle, $\theta$, and an azimuthal angle, $\phi$, are generated relative to the normal to the surface such that the distributions of $\cos^2 \theta$ and $\phi$ are uniform. If we let $p_1$ and $p_2$ denote two uncorrelated random variables with distributions that are uniform between 0 and 1, then we can let

$$\phi = 2\pi p_1$$

and

$$\theta = \cos^{-1} \left( \left[ 1 - p_2(1 - \cos^2 \theta_{\max}) \right]^{1/2} \right),$$

where $\theta_{\max}$ denotes a polar angle beyond which no trajectory that passes through $S_1$ can also pass through $S_2$. One then determines which of these trajectories pass through $S_2$ and rejects all the rest. If $n$ trajectories satisfy this criterion, then the geometrical factor to pass through both surfaces is

$$G = \frac{n}{N} \cdot \pi (1 - \cos^2 \theta_{\max}) A_1,$$

where $A_1$ is the area of $S_1$. Furthermore, the $n$ accepted trajectories correspond to a sample from the incident isotropic distribution that is unbiased other than by the condition that they pass through the two surfaces.

For some non-trivial detector geometries, such as that which is illustrated schematically in Figure 1, the fraction of acceptable trajectories, $n/N$, can be very small. Consequently it may be necessary to generate a very large number of trial trajectories that will subsequently be rejected because they do not pass through $S_2$. Several instruments for studies of solar energetic particles have been flown with detectors that are neither parallel nor coaxial [2, 3], as suggested in Figure 1.

An alternative approach to generating straight lines that pass through both $S_1$ and $S_2$ is to independently generate $N$ points with statistically uniform distributions on each of the surfaces and then connect corresponding points with straight lines. None of these trajectories would have to be rejected due to failure to meet the condition that they pass through both surfaces. However the distribution of incidence directions represented by these trajectories would not, in general, correspond to the isotropic distribution we want to sample.

In the following section we discuss a modification to this latter procedure that will produce the required angular distribution and also allow a simple calculation of the geometrical factor for trajectories to pass through $S_1$ and $S_2$. 

The desired geometric factor is approximated by
\[ dG_{1,2} = \frac{d\sigma_1 d\sigma_2}{r_{1,2}^2} (\hat{r}_{1,2} \cdot \hat{n}_1) (\hat{r}_{1,2} \cdot \hat{n}_2), \] (1)
where \( r_{1,2} \) is the magnitude of \( \vec{r}_{1,2} \) and \( \hat{r}_{1,2} \equiv \vec{r}_{1,2}/r_{1,2} \).

The geometrical factor for straight-line trajectories to pass through two finite, plane surfaces \( S_1 \) and \( S_2 \) having areas \( A_1 \) and \( A_2 \) can be obtained by dividing these areas up into infinitesimal sub-areas, expressing the contribution of each pair of these using the result from Equation 1, and integrating over the two areas:
\[ G = \int_{S_1} d\sigma_1 \int_{S_2} d\sigma_2 \frac{1}{r_{1,2}^2} (\hat{r}_{1,2} \cdot \hat{n}_1) (\hat{r}_{1,2} \cdot \hat{n}_2). \] (2)

This integral can be evaluated using Monte Carlo techniques as follows. A set of \( N \) points is generated with a uniform probability distribution over the surface \( S_1 \) and another, independent, uniformly-distributed set of \( N \) points is generated on \( S_2 \). (A procedure for generating such distributions on complex surfaces is discussed in the next section.) The desired geometric factor is approximated by
\[ G \simeq \frac{1}{N} \sum_{i=1}^{N} \frac{A_1 A_2}{N} \frac{1}{r_{1,2}^2} (\hat{r}_{i} \cdot \hat{n}_1) (\hat{r}_{i} \cdot \hat{n}_2) \]
(3)
\[ = \frac{A_1 A_2}{N} \sum_{i=1}^{N} w_i \]
(4)
with
\[ w_i \equiv \frac{1}{r_{1,2}^2} (\hat{r}_{i} \cdot \hat{n}_1) (\hat{r}_{i} \cdot \hat{n}_2). \] (5)
Here the sums are over the \( N \) vectors \( \vec{r}_{i} \) \((i = 1 \ldots N)\) connecting the \( i \)th point randomly generated on surface \( S_1 \) with the \( i \)th point randomly generated on surface \( S_2 \). The quantity \( w_i \) is a weighting factor for each of these vectors that takes into account the separation and the orientations of the two surfaces at the locations sampled by the vector.

The relative likelihoods of collecting particles with trajectories \( \vec{r}_{i} \) are just given by the weighting factors \( w_i \) (Eq. 5) used in calculating the geometrical factor defined by \( S_1 \) and \( S_2 \). Thus it is possible to produce a set of trajectories with the correct angular distribution by discarding some of the vectors on a statistical basis. If we use \( w_{\text{max}} \) to denote the maximum value among the \( N \) weights, we can generate a set of \( N \) independent random numbers \( p_i \) that are uniformly distributed between 0 and 1 and then retain only those trajectories that satisfy the condition
\[ p_i \leq \frac{w_i}{w_{\text{max}}}. \] (6)

The number of acceptable trajectories \( n \) that one obtains is subject to statistical fluctuations arising from the Monte Carlo process used to generate them. If one wants to obtain a specific number of trajectories, \( n' \), using this procedure, one can choose a value of \( N \) large enough such that the probability of getting \( \leq n' \) is essentially unity and then randomly discarding the extra \( n - n' \) trajectories. Alternatively, one can use the process to produce \( n \) trajectories, check whether \( n > n' \), and generate additional trajectories if it is not. With this latter procedure one needs to take into account the fact that the process for discarding trajectories described by Equation 6 depends on \( w_{\text{max}} \), the maximum weighting factor among all of the trial trajectories. So a correct procedure for adding additional trajectories is to combine the old with the new trial trajectories, find a new value of \( w_{\text{max}} \) considering the combined set, and then reject trajectories from that set using a uniform criterion. In cases where numbers of trial trajectories are large enough to guarantee that some trajectory will have a value of \( w_i \) close to the absolute maximum, this extra complication can typically be ignored.

The geometrical factor given by Equations 3–5 and the procedure described above for selecting a subset of trajectories that have the correct angular distribution apply not only for the case where surfaces \( S_1 \) and \( S_2 \) are planar. Curved, folded, or even discontinuous surfaces are acceptable provided that none of the trajectories originating on \( S_1 \) can pass through \( S_2 \) more than one time and vice versa.

### 3 Generating Uniformly Distributed Points

For a number of simple planer shapes (triangles, rectangles, circles, etc.) there exist analytic formulas that can be used to generate a random set of points with a statistically uniform density distribution over the surface based on two uniform random variables. For example, if \( \vec{x}_1, \vec{x}_2, \) and \( \vec{x}_3 \) denote denote vectors from the origin of a coordinate system to three non-colinear points in three-dimensional space, one can generate a uniform distribution of points, \( \vec{x} \), on the triangular area having \( x_1, x_2, \) and \( x_3 \) as vertices using the formula
\[ \vec{x} = \vec{x}_1 (1 - p_1) p_2^{1/2} + \vec{x}_2 p_1^{1/2} + \vec{x}_3 \left(1 - p_2^{1/2}\right), \] (7)
where \( p_1 \) and \( p_2 \) are two independent random variables uniformly distributed between 0 and 1.

For more complicated shapes one can break the figure up into sub-regions for which analytic formulas exist and generate points on each of them such that the probability...
of a given point falling in a particular sub-region is proportional to its area.

It is often sufficient to replace a shape of interest with a polygon, allowing the number of sides to be large enough to adequately approximate the original shape. In geometrical factor calculations the approximating polygon should normally be required to have the same area as the original figure. For generating a statistically uniform distribution of points over the surface of a polygon one can subdivide the polygon into a set of triangles and use the formula in Equation 7 to distribute points on each of the triangles.

The problem of subdividing a simple polygon (i.e., a polygon with no holes) into a set of triangles is well studied and a number of suitable algorithms are available (see, for example, [4, 5]). For the case where the polygon is convex, the triangulation process is particularly simple: one can just cut the polygon along line segments connecting one selected vertex to each of the other vertices.

4 Non-isotropic Angular Distributions

The geometrical factor, $G$, is normally thought of as the ratio between the intensity of incident particles and the counting rate of coincidences they produce between the two detector elements represented by the surfaces $S_1$ and $S_2$. This interpretation depends on the assumption that the particle distribution being measured is isotropic. If, on the other hand, one has a particle distribution that depends on viewing direction as $F(\hat{r})$ (representing the intensity of particles per unit area and unit solid angle with velocities directed opposite to $\hat{r}$), one needs an expression for the "directional response" of the instrument, $A(\hat{r})$, so that the counting rate will be the integral of $F(\hat{r})A(\hat{r})$ over all solid angles, $\Omega$. Using the notation from Equation 2 we can write

$$\int_{\Omega} d\Omega F(\hat{r})A(\hat{r}) = \int_{S_1} d\sigma_1 \int_{S_2} d\sigma_2 \frac{F(\hat{r}_{1,2})}{r_{1,2}^2} (\hat{r}_{1,2} \cdot \hat{n}_1) (\hat{r}_{1,2} \cdot \hat{n}_2).$$

The Monte Carlo approach can be used to evaluate this integral as was done in the case of an isotropic population of incident particles with the result

$$\int_{\Omega} d\Omega F(\hat{r})A(\hat{r}) \simeq \sum_{i=1}^{N} \frac{A_1 A_2}{N} F(\hat{r}_i) \frac{1}{r_i^2} (\hat{r}_i \cdot \hat{n}_1) (\hat{r}_i \cdot \hat{n}_2)$$

$$= \frac{A_1 A_2}{N} \sum_{i=1}^{N} w_i$$

with

$$w_i = \frac{F(\hat{r}_i)}{r_i^2} (\hat{r}_i \cdot \hat{n}_1) (\hat{r}_i \cdot \hat{n}_2).$$

The only change in the calculation is that now the incident particle intensity distribution function appears in the weighting factor that goes into the summation and must now be taken into account when rejecting trajectories based on Equation 6 in order to obtain a set having the correct angular distribution.

5 Summary

We have presented a procedure for evaluating the geometrical factor for trajectories drawn from an isotropic distribution of incidence directions to pass through two surfaces. A set of trial incidence directions are generated as lines connecting sets of points having statistically uniform distributions on the two surfaces. For each direction a weighting factor is calculated based on the separation between the points where it intersects the two surfaces and on the angles it makes with the vectors normal to the surfaces. The geometrical factor is proportional to a sum over these weighting factors. A subset of the directions that correctly represents the underlying isotropic distribution is obtained by statistically discarding a fraction of the trajectories based on their weighting factors. A generalization to the case of an arbitrary distribution of incidence directions was also described.

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References