On the modulation of cosmic rays as described by a stochastic transport model

R.D. Strauss¹, M.S. Potgieter¹, A. Kopp², I. Büsching³

¹Centre for Space Research, North-West University, Potchefstroom, 2520, South Africa
²Institut für Experimentelle und Angewandte Physik, Christian-Albrechts-Universität zu Kiel, Leibnizstraße 11, 24118 Kiel, Germany
³Institut für Theoretische Physik, Lehrstuhl IV: Weltraum- und Astrophysik, Ruhr-Universität Bochum, 44780 Bochum, Germany

dutoit.strauss@nwu.ac.za

Abstract: We present selected results from a newly developed 5 dimensional cosmic ray (CR) modulation model, employing stochastic differential equations to solve the relevant transport equation. The model is applied to the study of galactic CR electron and proton modulation. We calculate the propagation times and energy losses suffered by these particles as they propagate through the heliosphere, and compare these values with the limited analytical approximations that are available in order to validate the approach. We also incorporate a wavy current sheet into the model and examine the effect thereof. We argue that the stochastic modelling approach gives additional insight into the modulation processes, with respect to what is found with traditional finite difference modulation models.

Keywords: cosmic rays, heliosphere, modulation, energy losses, propagation times, numerical methods

1 Introduction

The propagation of cosmic rays (CRs) through the heliosphere is described by the well-known Parker [5] transport equation (TPE)

\[ \frac{\partial f}{\partial t} = \left( -\vec{V}_{sw} + \langle \vec{v}_d \rangle \right) \cdot \nabla f + \nabla \cdot (K_s \cdot \nabla f) + \frac{P}{3} \left( \nabla \cdot \vec{V}_{sw} \right) \frac{\partial f}{\partial P}, \]

related to the differential intensity by \( j = P^2 f \), which contains all the relevant transport processes. Normally, Eq. (1) is solved numerically to obtain the CR differential intensity \( j \); mostly using finite differences schemes to do so. Recently, the focus has shifted to using alternative numerical schemes, with numerical solutions of stochastic differential equations (SDEs) widely implemented. With this approach, not only can \( j \) be calculated, but also additional transport quantities, most notable the CR propagation times and energy losses. In this work, and an accompanying paper ([10]), we present selected results from our newly developed 5D SDE model, while also demonstrating the versatility of this approach in calculating these additional transport quantities, thereby leading to additional insight into the modulation process.

2 The Stochastic Transport Model

By applying Itô’s lemma (see e.g. [2, 3]) to the TPE, the applicable set of SDEs governing CR transport can readily be obtained

\[ dr = \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 V_{sw} \right) + \frac{1}{r \sin \theta} \frac{\partial \delta r \phi}{\partial \phi} - V_{sw} - v_d \right] ds \]

\[ + \sqrt{\frac{2}{\kappa_{rr}}} dW_r + \sqrt{\frac{2}{\kappa_{\phi \phi}}} dW_\phi \]

\[ d\theta = \left[ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \delta \theta \phi \right) - \frac{v_d \phi}{r} \right] ds + \sqrt{\frac{2}{\kappa_{\theta \theta}}} r dW_\theta \]

\[ d\phi = \left[ \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( r \delta \phi \phi \right) - \frac{v_d \phi}{r \sin \theta} \right] ds + \sqrt{\frac{2}{\kappa_{\phi \phi}}} r \sin \theta dW_\phi \]

\[ dE = \left[ \frac{1}{3r^2} \frac{\partial}{\partial r} \left( r^2 V_{sw} \right) \Gamma E \right] ds \]

in terms of backwards time \( s \) (see also [7, 8]). The energy SDE can also be rewritten in terms of rigidity as

\[ dP = \left[ \frac{1}{3r^2} \frac{\partial}{\partial r} \left( r^2 V_{sw} \right) P \right] ds, \]

with
\[ \Gamma = \frac{E + 2E_0}{E + E_0} \]  

(4)

As the time backwards method is applied to solve the set of SDEs, an initial phase-space point is chosen, \( x_i^0, i \in \{r, \theta, \phi, E\} \) at time \( s = 0 \), which is also the point at which \( j \) will be obtained. The evolution of this phase-space point (pseudo-particle) is then calculated iteratively according to the set of SDEs

\[ x_i^{j+1} = x_i^j + \Delta x_i^j \]  

(5)

until a modulation boundary is reached (for galactic CRs, the heliopause) at time \( s = \epsilon \). Doing this for a large number of these particles, an average value of \( j \) is obtained. Moreover, the CR propagation time can be calculated directly from the numerical scheme as

\[ \tau = |s^e - s^0|. \]  

(6)

We however calculate \( \tau \) for a large number of particles and examine the behaviour of this average, labelled as \( \langle \tau \rangle \). Similarly, the average energy loss \( \langle \Delta E \rangle \) can also be calculated by averaging \( \Delta E = E^e - E^0 \) for a large number of particles. Here \( E^e \) is the energy with which the particle enters the heliosphere, while \( E^0 \) is the energy at which it is observed at some point, say Earth.

3 Selected Results

Fig. 1 shows pseudo-particle traces of galactic electrons reaching Earth for the \( A > 0 \) (left panel) and \( A < 0 \) (right panel) heliospheric magnetic field (HMF) cycles. This illustrates the additional insight gained from using the SDE approach: Fig. 1 shows the characteristic of electron drifts, with the particles reaching Earth originating and staying near the equatorial regions in the \( A > 0 \) cycle, while drifting mainly from the polar regions in the \( A < 0 \) cycle.

\[ \langle \tau_{cd} \rangle = \left[ \frac{1}{\langle \tau_d \rangle} - \frac{1}{\langle \tau_c \rangle} \right]^{-1} \]  

(7)

The validity of the SDE approach in calculating \( j \) was shown previously by various benchmarking studies. As an example of energy spectra calculated by the SDE approach, Fig. 2 shows galactic electron spectra, at Earth, for various drift conditions as calculated by the spatially 3D model. These results are both qualitatively and quantitatively similar to solutions using finite differences schemes, but with the latter approach unable to calculate \( \tau \) and \( \Delta E \).

Using a spatially 1D version of the SDE modulation model, Fig. 4 shows calculations of \( \langle \tau \rangle \) (left panel) and \( \langle \Delta E \rangle \) (right panel) as the scatter points, as a function of the 1D diffusion coefficient. The SDE results compare well with the analytical solution of

Figure 1: Pseudo-particle traces for 100 MeV galactic electrons for the \( A > 0 \) (left panel) and \( A < 0 \) (right panel) HMF cycles. Taken from [8].

Figure 2: Galactic electron spectra at Earth for various drift conditions (taken from [8]).
for the average propagation time, with the characteristic convection and diffusion time-scales given by

$$\langle \tau_c \rangle = \frac{R}{V_{sw}} \quad \text{and} \quad \langle \tau_d \rangle = \frac{R^2}{6 \kappa}. \quad (8)$$

The calculated energy loss compares well with the analytical solution of [6], given by

$$\left( \frac{E}{E_0} \right) = \left( 1 - \frac{\Gamma R V_{sw}}{3 \kappa} \right)^{-1} \quad \text{(9)}$$

and valid for $\kappa \gg 1$. Note that two solutions for $\Delta E$ are given, namely for totally relativistic CR ($\Gamma = 1$, i.e. for galactic electrons which are highly relativistic for all energies considered) and for non-relativistic CRs ($\Gamma = 2$). The SDE proton results should be somewhere in between these two extreme cases and indeed they are.

Fig. 3 shows $\langle \tau \rangle$ and $\langle \Delta E \rangle$, calculated for galactic protons (top panels) and electrons (bottom panels) for the $A > 0$ (left panels) and $A < 0$ (right panels) drift cycles using the spatially 3D SDE model. Black fills indicate $\langle \tau \rangle$, while grey fills indicate $\langle \Delta E \rangle$. Taken from [9].

Due to the dependence on $\Gamma$, protons will generally loose energy faster than the electrons. Electrons on the other hand have a significantly larger mean free path.
Figure 4: Calculations of $\langle \tau \rangle$ (left panel) and $\langle \Delta E \rangle$ (right panel) using a spatially 1D version of the SDE modulation model.

MeV, the energy losses suffered by the electrons and protons are very similar. This is in contrast to the generally held belief that electrons suffer little or no energy losses and is discussed in detail in [9]. We also note that $\langle \Delta E \rangle$ is larger for the drift cycle in which $\langle \tau \rangle$ is largest. The decrease of $\langle \Delta E \rangle$ for electrons below $\sim 100$ MeV is due to our choice of an energy independent diffusion tensor for electrons below 1 GV.

4 Discussion

We have presented selected results from a newly developed 5D SDE modulation model. The calculated propagation times and energy losses were benchmarked with existing analytical approximations in 1D, whereafter these quantities were calculated with the spatially 3D modulation model that includes CR drifts. We can conclude that:

- For the spatially 1D scenario, $\langle \tau \rangle$, as calculated by the SDE model, compares well with the probability wave approach of [5]. The results suggest that the calculations of [4] are insufficient to describe the behaviour of $\langle \tau \rangle$ at small values of $\kappa$. Here, convection dominates the diffusive process, so that CRs are unable to enter the heliosphere resulting in $\langle \tau \rangle \to \infty$.

- The SDE calculations of $\langle \Delta E \rangle$ also compare well with the available 1D analytical approximations. Similar to $\langle \tau \rangle$, $\langle \Delta E \rangle$ increases with decreasing values of $\kappa$ as the CRs generally spend more time in the supersonic solar wind and therefore loose more energy adiabatically. We however note that: (i) Although $\langle \tau \rangle$ gives some indication of $\langle \Delta E \rangle$, there is a non-linear relationship between these two quantities and one is unable to state that $\langle \tau \rangle \propto \langle \Delta E \rangle$. (ii) These calculations (and subsequent interpretation) are only valid in the supersonic solar wind region where $\nabla \cdot \vec{V}_{sw} > 0$. If however $\nabla \cdot \vec{V}_{sw} = 0$ (as might occur in certain regions in the heliosheath) CRs cannot lose any energy, even when $\langle \tau \rangle \to \infty$.

- For the spatially 3D scenario, both $\langle \tau \rangle$ and $\langle \Delta E \rangle$ are larger for the drift scenario (for electrons and protons) in which the CRs drift inwards to Earth along the neutral sheet.

- Above $\sim 1$ GV, electrons spend more time in the heliosphere than protons at the same energy, but loose less energy. This again emphasizes the non-linearity between these quantities.

- By demonstrating the ability and effectiveness of the SDE approach in calculating $\langle \tau \rangle$ and $\langle \Delta E \rangle$, we argue that this modelling approach can provide additional insight into the CR modulation process.

References