Acceleration of energetic particles by compressible plasma waves of arbitrary scale sizes

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Abstract: This paper presents a calculation of energetic particle acceleration by a train of compressible plasma waves of arbitrary scale sizes. It is an extension of the work by Zhang [1]. The compressible plasma waves can broaden the momentum distribution of particles significantly only when the particle diffusion scale \( \kappa/V \) is greater than the size of each compression zone but is smaller than the spatial extent of wave presence. The accelerated particles downstream of the compressive waves will typically have a power-law distribution at momenta far away from the source injection momentum with a slope only slightly steeper than the diffusive shock acceleration with the same plasma compression ratio. If the source particles are continuously injected in the entire wave span, after enough wave cycles, the summed distribution of accelerated particles can approach to an asymptotic \( p^{-3} \) distribution which does not seem to depend on any model parameters. This distribution of particles contains divergent amount of pressure. A nonlinear interaction between the wave and accelerated particles must be considered in order to produce a \( p^{-5} \) distribution.

Keywords: Acceleration of particles — Diffusion — plasma waves — cosmic rays

1 Introduction

Compression of plasma is a ubiquitous phenomenon in space. It occurs in the solar wind when fast streams overtake preceding slow streams. Examples are corotating interaction region (CIR) and global merged interaction region (GMIR). On small scale there is considerable compressible turbulence in the solar wind flow during quiet time [2]. Compressible waves can be generated from Alfvén waves at shock front. Interstellar turbulence can also be strongly compressible, as indicated by observation of supersonic flow of interstellar medium [3]. Star formation starts from the dense seeds produced by compression of interstellar medium. When the stellar wind interacts with interstellar medium or the wind from adjacent stars, compression of plasma occurs.

Cosmic rays or energetic particles are scattered by fluctuating medium with embedded magnetic fields. During the scattering, particles can gain or lose energy. Since the particle gain or loss of energy in turbulence is completely random, we often call it stochastic acceleration. There are a large number of litterateurs dealing with the stochastic acceleration of particles (see review [4]). Recently, there is a renewed interest in this acceleration mechanism as Fisk et al. [5, 6] suggested it can be used to explain a seemingly universal \( f \propto p^{-5} \) distribution observed in the heliosphere [7], while the study by Jokipii and Lee [8] argued it cannot guarantee the \(-5\) power law index. The stochastic acceleration is typically described as diffusion in energy or magnitude of momentum space. It is a natural result coming out of the quasi-linear approximation used by most authors. However, in many cases, compression of plasma in interplanetary or interstellar medium is not small in both magnitudes and spatial scales. In this limit, perturbation method of quasi-linear theory may no longer be valid. The work presented in this paper considers particle acceleration by compressive plasma structure of arbitrary scale sizes. This is an extension of the work by Zhang [1] where the compressive wave is approximately by series of compressional shock waves and rarefaction regions. It was found that compressional particle acceleration can only lead to a universal \( f \propto p^{-3} \) distribution, but if non-linear wave-particle interaction is considered, the effect of pressure from the accelerated particles on the wave can eventually lead to a \( f \propto p^{-5} \) distribution.

2 Model

The isotropic part of particle distribution function as a function of particle momentum in plasma reference frame \( p \), position \( x \) and time \( t \) follows a Fokker-Planck diffusion equation:

\[
\frac{\partial f}{\partial t} = \nabla \cdot \kappa \cdot \nabla f - \nabla \cdot \nabla f + \frac{1}{3} (\nabla \cdot \nabla) p \frac{\partial f}{\partial p} \quad (1)
\]

where \( \kappa \) is the particle diffusion tensor, \( \nabla \) the background plasma velocity. In this paper, we consider particle acceleration by a train of \( N \) compressible plasma wave cycles with gradually varying plasma speed which has a gradient-
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Figure 1: Profile of the plasma speed and particle diffusion coefficient used in the model calculation.

t scale size comparable to or larger than the diffusion s-scale κ/V. For simplicity, we consider a steady state 1-d case in which the system varies only with spatial x-axis as well as the particle momentum p. In order to obtain analytical solution to the transport equation, we prescribe a saw-tooth profile for the plasma speed as shown in Figure 1. The plasma speed is relative to the reference frame of the wave, in which the structure is stationary. Upstream or downstream of the wave train, the plasma has a constant speed \(V_1\). The plasma in the wave train is periodically compressed or rarefied. In each wave cycle, the plasma is compressed when its speed drops and it expands when its speed increases. The saw-tooth profile means that the speed varies linearly with the distance in the each of the compression or rarefaction regions. The saw-tooth can be asymmetric, depending on the ratio of \(x_2/x_1\). The ratio of peak to bottom speed, \(V_1/V_2\), is the amplitude of plasma compression ratio. In order get the analytical solution, we also assume that the diffusion coefficient is proportional to the plasma speed square throughout the entire volume and it is independent of particle momentum.

A source of mono-energetic particles is injected upstream with distribution \(f_0(p) = \delta(p - p_0)/p_0^2\). We first look at how the particle distribution evolves as it through repeated acceleration and cooling. Then we calculate the particle spectrum if the particles are continuously injected into the waves.

3 Result

With the above model input, the transport equation (1) can be solved with the method of Mellin transform (see e.g., http://en.wikipedia.org/wiki/Mellin_transform). The particle distribution function downstream of \(n\) wave cycles can be written as a summation of series of power laws

\[
f_n(p) = \begin{cases} 
- \frac{1}{p_0^2} \sum_{i=1}^{\infty} \text{Res}(\chi(\gamma_i)) \left( \frac{p}{p_0} \right)^{-\gamma_i} & p > p_0 \\
\frac{1}{p_0^2} \sum_{i=1}^{\infty} \text{Res}(\chi(-\gamma_i)) \left( \frac{p}{p_0} \right)^{-\gamma_i} & p < p_0 
\end{cases}
\]

where \(\gamma_i\) is the \(i\)-th pole on the positive real axis with its residue \(\text{Res}(\chi(\gamma_i))\) and \(-\gamma_i\) is the \(i\)-th pole on the negative real axis with its residue \(\text{Res}(\chi(-\gamma_i))\) of the momentum transformation function:

\[
\chi(\gamma) = -\frac{\gamma_i \left[ \text{det}(A_2) \text{det}(A_4) \right]^n}{\left[ \text{det}(A_1) \text{det}(A_3) \right]^n (B^n)_{21} + r_1 (B^n)_{22}}
\]

\[
A_1 = \begin{bmatrix} -\frac{1}{s_1} & -\frac{1}{s_2} \\ \frac{1}{s_1} & \frac{1}{s_2} \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} \frac{V_2}{x_2} & \frac{V_s}{x_2} \\ -\frac{1}{s_1} & -\frac{1}{s_2} \end{bmatrix}
\]

\[
A_3 = \begin{bmatrix} -\frac{V_2}{x_2} & \frac{V_s}{x_2} \\ -\frac{s_1}{s_2} x_1 & \frac{1}{x_2} \end{bmatrix}
\]

\[
A_4 = \begin{bmatrix} \frac{V_s}{x_2} & \frac{V_s}{x_2} \\ \frac{s_1}{s_2} x_1 & \frac{1}{x_2} \end{bmatrix}
\]

with exponents:

\[
s_{1,2} = \frac{-3(1 + r_1) \pm [9(1 + r_1)^2 - 12r_1]^{1/2}}{6}
\]

\[
s_{3,4} = \frac{-3(1 - r_2) \pm [9(1 - r_2)^2 + 12r_2]^{1/2}}{6}
\]

The parameters are

\[
r_1 = \frac{V_1 x_1}{r_1} V_1 \text{ and } r_2 = \frac{x_2}{x_1} r_1
\]

which represent the ratio of the size of the compression or expansion zone to the particle diffusion length. \(B^n\) is the \(n\)-th power of the matrix \(B\) which is a product of four matrices \(B = A_4 A_3^{-1} A_2 A_1^{-1}\). The calculation of \(B^n\) can be easily achieved with the eigen-decomposition method. \((B^n)_{21}\) and \((B^n)_{22}\) are the two elements in the second row of the \(B^n\) matrix.

Figure 2 shows a calculation of downstream particle distribution function \(f_n(p)\) as a function of particle momentum \(p\). The compression amplitude used in the calculation is 4. The plasma speed profile is a symmetric saw-tooth, meaning that the sizes of the compression zone and rarefaction region are the same. After acceleration and cooling by 1 wave cycle, the particle distribution is very much a power-law on the both sides of \(p/p_0 = 1\). This is because the second power indices are much greater than first indices. The slope of particle distribution function \(f\) as a function particle momentum \(p\) is \(-\gamma_i\) on the \(p > p_0\) side and its slope on the \(p < p_0\) side is \(-\gamma_i\). Even for a case of symmetric wave, in which the compression and expansion
Far away from \( p_0 \), either \( p \gg p_0 \) or \( p \ll p_0 \), the distribution function is a power law dominated by the first power-law with slope \( -\gamma_1 \) on the \( p > p_0 \) side while the slope on the \( p < p_0 \) side is \( \gamma_1 \). The values of \( \gamma_1 \) and \( \gamma_1 \) change only slightly with the number of waves \( n \). Figure 3 shows the slopes as a function of \( \kappa_1/V_1x_1 \) for various numbers of wave cycles. First, let’s look at the slope after 1 cycle of acceleration and cooling. For particles with a small diffusion coefficient or \( \kappa_1/V_1x_1 \ll 1 \), both the negative slope \( \gamma_1 \) on the \( p > p_0 \) side and positive slope \( \gamma_1 \) on the \( p < p_0 \) side are large, indicating that the distribution function is narrowly peaked at \( p_0 \). It means that the particles get little acceleration or cooling after a complete cycle and their momentum returns to the original value with little change. Similarly for particles that have a too large diffusion coefficient or \( \kappa_1/V_1x_1 \gg 1 \), the slopes of the distribution function on the both sides of \( p_0 \) are steep, indicating little acceleration or cooling. The slopes \( \gamma_1 \) and \( \gamma_1 \) are near minimum or the distribution function is the fattest when \( \kappa_1/V_1x_1 \sim 1 \). This suggest that best efficiency of acceleration or cooling occurs when the scale sizes of compression zone and rarification region are comparable to particle diffusion length.

When diffusion scale is too small, particles just undergo adiabatic heating and cooling and the system returns to its original state after a complete cycle. If the diffusion is too large, the particles can see both the compression zone and rarification region simultaneously and acceleration and cooling effects cancel each other quickly.

As \( n \) increases, the slopes \( \gamma_1 \) and \( \gamma_1 \) decrease in the region where \( \kappa_1/V_1x_1 \) is approximately greater than 1, forming a broader minimum over a wide range of \( \kappa_1/V_1x_1 \). The minimum slope for the distribution function \( f \) on the \( p \gg p_0 \) side is \( -\gamma_1 = -4.95 \), while the slope on the \( p \ll p_0 \) side is \( \gamma_1 = 1.95 \). The condition for reaching the minimum spectral slope is \( 1 < \kappa_1/V_1x_1 < N \). If we think that the change of diffusion coefficient \( \kappa_1 \) comes from the change of particle energy, a power-law spectrum of the minimum slope is maintained for all particles energies as long as the condition of \( 1 < \kappa_1/V_1x_1 < N \) is satisfied. The physical requirement for particle get enough acceleration from compressible plasma wave trains is that the particle diffusion length is greater than the size of the individual of compression zone but less than the size of entire wave train. All particles with a diffusion length satisfying the inequality can achieve the same power law distribution.

The first positive and negative poles of \( \chi(\gamma) \) are situated in a region where the matrix \( B \) has two real eigen values slightly above 1 for large enough \( \kappa_1/V_1x_1 \). As \( n \to \infty \), the matrix product \( B^n \) is dominated by the larger of the two eigen values. In this limit, we derive a formula for the first positive and negative pole locations

\[
\gamma_1 = \frac{3[R - 1 - \ln(R)]}{R - R^{-1} - 2 \ln(R)} \quad \text{and} \quad \gamma_1 = \gamma_1 - 3 \quad (11)
\]

where \( R = V_1/V_2 \) is the amplitude of the plasma compression ratio. These asymptotic minimum slopes do not depend on \( x_2/x_1 \) and \( \kappa_1 \) values as long as the minimum slope condition \( 1 < \kappa_1/V_1x_1 < N \) is met. If we...
Plug in the a compression ratio \( R = V_1/V_2 = 4 \), we get \( \gamma_1 = 4.9529 \cdots \) and \( \gamma_1 = 1.9529 \cdots \). For comparison, the spectral slope of particles by diffusive shock acceleration is \( \gamma_{sh} = 3R/(R - 1) \). When \( R = 4 \), \( \gamma_{sh} = 4 \). The spectral slope produced of compressible waves is slightly steeper than the shock with the same compression ratio. The shock can only have a maximum compression ratio of 4, but continuous plasma compression may go beyond it although it is very rare to see such a strong compressible plasma wave. From this calculation, we can say that particle acceleration by compressible plasma waves is very much like diffusive shock acceleration. Particles can gain or lose energy by jumping between compression zone and rarefaction region even for particles with large mean free paths compared to the wavelength. Whether a particle gains or lose energy depends purely on statistics. It turns out the probability distribution for a particle to gain certain amount of energy is very similar to that at a shock.

The above calculation of particle distribution function is for particles initially injected upstream of the wave train. Sometimes, particles are continuously injected in the entire region where the waves travel through. This case can be modeled with particle injection occurring throughout all the wave cycles. The particle distribution function is a summation of all those particles injected, i.e.,

\[
N(p) = \sum_{n=1}^{N} f_n(p)
\]

where \( f_n(p) \) comes from (2). All these calculations start with mono-energetic injection. Figure 4 shows how the summed distribution function evolves with the number of wave cycles \( N \). After one wave cycle, the distribution function becomes an approximate double power-law with a slope \( \gamma_{p} \) on the \( p < p_0 \) side and a slope \( -\gamma_{p} \) on the \( p > p_0 \) side. As the number of wave cycles increases, the slopes of the distribution function on both sides of \( p/p_0 = 1 \) become less steep. The distribution above \( p_0 \) starts approximately with a power-law until some cut-off momentum. Eventually, after a sufficient number of wave cycles, the distribution function approaches to a flat distribution on the \( p < p_0 \) side and \( f \propto p^{-3} \) power-law distribution on the \( p > p_0 \) side, first starting at low momentum and then propagating to higher and higher momentum. This asymptotic distribution is same as the asymptotic distribution found by [1] with a prescribed square wave for the plasma speed profile. It seems that the asymptotic distribution does not depend on any model parameter. The plasma compression ratio, the size of compression or rarefaction region, particle diffusion coefficient and the plasma speed profile inside the wave only determine how fast the asymptotic distribution will be approached. Because of limited space here, the presentation of these cases has been left out.

4 Summary

We have shown that the behavior of particle acceleration by compressible plasma waves of arbitrary compression and rarefaction sizes are similar to those waves consisting of series compressional shocks in [1]. If only particle acceleration is considered, the acceleration of multiple waves can produce a power-law distribution only slightly steeper than a single shock with an same compression ratio. If particles are continuously injected over the entire wave span, the summed particle distribution can approach to a \( p^{-3} \) universal distribution. Since this distribution contains a divergent amount of pressure from the accelerated particles, non-linear interaction must occur to limit the rate of acceleration. A possible mechanism to produce a \( p^{-3} \) distribution through a balance between the compressional particle acceleration and particle loss mechanisms such as large-scale adiabatic cooling or particle escaping has been given in [1].

References
